(*Calculations for Lemma 4.2 Case 1*)

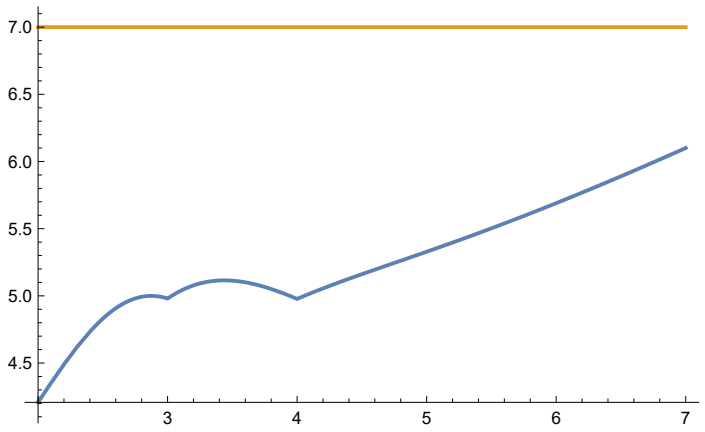
In[•]:=
```
Simplify[
 (1/(2 + p)) * (1/(2 + p)) * (p/(2 + p))^p + (1/(2 + p)) * (1/(2 + p))^p * (p/(2 + p)) + (1/(2 + p))^p * (1/(2 + p)) * (p/(2 + p))]
```

Out[•]=

$$\frac{2\,p\left(\frac{1}{2+p}\right)^p+\left(\frac{p}{2+p}\right)^p}{(2+p)^2}$$

In[•]:=
```
Plot[{((3/(2 * E)) * Max[{1/2^(p - 1), 1/2^3 ((p - 1)/p)^(p - 4)}] + Max[{1/2^p, 1/2^3 (p/(p + 1))^(p - 3)}])/
  ((p + 2) * (2 p (1/(2 + p))^p + (p/(2 + p))^p)/(2 + p)^2), 7}, {p, 2, 7}]
```

Out[•]=



In[•]:=
```
NMaximize[((3/(2 * E)) * Max[{1/2^(p - 1), 1/2^3 ((p - 1)/p)^(p - 4)}] + Max[{1/2^p, 1/2^3 (p/(p + 1))^(p - 3)}])/
  ((p + 2) * (2 p (1/(2 + p))^p + (p/(2 + p))^p)/(2 + p)^2), p ≥ 2 && p ≤ 7, {p}]
```

Out[•]=

{6.10037, {p → 7.}}

(*Calculations for Lemma 4.2 Case 2*)

In[•]:=
```
Block[{p = 8}, Simplify[(3 E)/2 * ((p - 1)^2 + 3)/((p - 1)^2 - 1) ((p + 2) (p - 3))/(p - 2)^p + (p + 2)/(p - 2) * 1/E) +
  (2 - p + p^2)/((-2 + p) (1 + p)) ((p + 2) (p - 2))/(p - 1)^(p + 1) + (p + 2)/(p - 1) * 1/E) * E^2]]
```

Out[•]=

$$\frac{65}{24}+\frac{72\,163\,555\,e}{47\,029\,248}+\frac{580\,e^2}{363\,182\,463}$$

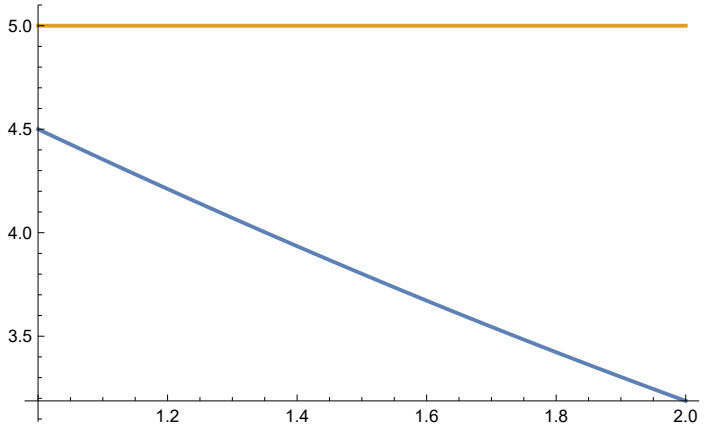
In[•]:=
$$N\left[\frac{65}{24}+\frac{72\,163\,555\,e}{47\,029\,248}+\frac{580\,e^2}{363\,182\,463}\right]$$

Out[•]=
6.87939

(*Calculations for Lemma 4.4 Base case*)

In[•]:=
$$\text{Plot}\left[\left\{\frac{p\left(\frac{1}{2}\right)^{2-p}\left(\frac{1}{9}\right)^{p-1}+\frac{1}{4^{p-1}}}{(p+2)\left(\frac{1}{3}\right)^{p+1}},\,5\right\},\,\{p,\,1,\,2\}\right]$$

Out[•]=



In[•]:=
$$\text{NMaximize}\left[\frac{p\left(\frac{1}{2}\right)^{2-p}\left(\frac{1}{9}\right)^{p-1}+\frac{1}{4^{p-1}}}{(p+2)\left(\frac{1}{3}\right)^{p+1}},\,p\geq 1\,\&\&\,p\leq 2,\,\{p\}\right]$$

Out[•]=
{4.5, {p → 1.}}